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Mathematics: A Knot of Languages. Variations on a Theme by Bentham

Jean-Pierre CLÉRO[1]

[1] Emeritus Professor at the University of Rouen, France; Lecturer at Scineces Po, Paris, France. jp.clero@orange.fr

Abstract: « 2 + 2 = 4 is a question of language. »
Bentham J., *Rationale of judicial evidence*, London, Hunt & Clarke, 1827, Vol. III, p. 274.
« The beginning and the end are never the most interesting […]. Only the middle is interesting. […]. We start again from the middle. French people think too much in terms of tree: the tree of knowledge, tree points, alpha and omega, roots and summit. It is the contrary of a blade of grass. Not only grass grows in the middle of things, but it grows itself from the middle. […] Grass has a line of flight but it has no rooting. We have grass in our head; not a tree: that is the meaning of what thinking is; what is a brain: «a certain nervous system» of grass ».
Deleuze G. & Parnet C., *Dialogues*, Champs Essais, 1966, p. 50-51.

Keywords: *mathematics; Knot of Languages; Variations; Bentham.*

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It is known that Bentham defined *fiction* as a language being of which the object was attributed existence by the author or the user of it. Most of the time, fiction is obtained by nouning a process, i.e. by attributing a name to a process, which is presented, in spite of its fluctuation, as a thing or as objective. So, it is supposed that, to make the fiction work, it is necessary that some beings or acts are not considered fictitious, but rather are real and that the fictions are as if they were superstructures built upon a real base or on a base pretended to be real. The strata which are built up in each superstructure gradually recede from the basement held as if it were in contact of things. It is also known that, in each area inspected, law, ethics, religions, sciences, fictional functioning are observed. Mathematics are no exception: in *Chrestomathia*, that is a book that investigates what would be useful to be learned and taught to the members of a modern society for itself and for each of them, we are witnessing, particularly in Appendix n° VIII, an analysis of this discipline from this point of view. How to do, in order to teach the very diverse intelligences that make up a society -which cannot be isolated from the other societies- disciplines that suit a modern society at the level that suits each individual?

So, mathematics is scrutinized for usefulness, in very different and intermingled senses. What is useful to be learned in a society is not necessarily the same thing for each other. And yet, in order to say that a learning is more useful than another, it is necessary that these utilities be commensurable. The Appendix n° VIII views this commensurability neither in the community of some character of human nature nor in different psychological characters that end up combining with each other; but it views it in the same belonging to language; like law, ethics, religion and the other disciplines, mathematics offers a wide variety of uses, included inside itself. Arithmetic, algebra, geometry, topology, the calculation of probabilities - whatever subjective probabilities or objective probabilities- require very different bundles of language uses, even if they can be considered as the arborescence of the same tree or of the same grove. Whatever society of which they are citizens, men share mathematics as they share language, considered in its various tongues and in its various uses within each of them.

Properly speaking, mathematics has no self-existing objects that would allow everyone access, however diverse, like perception seems to relate to specific objects -if the objects of perception exist in themselves in the way of given things without language. Its objects are not distinct and independent from what may be laid by the speech acts like definitions, axioms, reasonings and demonstrations. Therefore, the fictitious nature of these objects appears from the outset, even if we deal with objects that seem

to have a sort of substantiality or particularly tough stability by themselves, like numbers, points, lines, areas, volumes, and the acts that allow to relate to those objects. Everybody knows that such objects do not exist like minute color spots, like edges of objects or like traces that would be continuous as they seem to be. Therefore, they can be taken as fictions; but among those fictions, some are, in a way, more so than others. We will particularly focus on the latter that are fictions more than others. It is obviously not a question of listing them all; the matter will be to distinguish and to classify them, that is to say to group them into sections: a fluxion or an infinitesimal are not fictions as a transcendent number that is however divisible to infinity. A fraction of certainty is not calculated as a quantity that may be added and will then make disappear, in order to obtain a quadrature, rectification or center of gravity.

However, the idea that appeared as essential in Appendix n° VIII of *Chrestomathia* on the question of fictions is less that of their classification than that of the interleaving of the sorts of discourses that mathematics bring into play in order to pretend, in some way, to have an object; and especially, in spite of the fallacious word taken over from the *Essay on the History of Civil Society* of A. Ferguson, the way in which the diverse discourses are linked together such as any of them, whatever it may be symbolic -and it may be so in several modes- or more vulgar and of vernacular use, whatever it may be verbal (as in the preceding cases) or more intuitive, be alternately fundamental in the function of a base, or supported by that base. It might be that the most important fiction introduced to the field of mathematics by *Chrestomathia* - so that it can disseminate in other areas, through this structure which allows three uses of language -three languages perhaps- to coexist without one definitely dominating the others but in such a way that we have to try the whole set of languages as you try a set of keys. The languages in mathematics might be linked like Borromean rings so that you could not change any single one -and specifically break one- without changing the others -and particularly without freeing them from each other.

I. Fictitious objects and acts favored by fictions

If « a fictitious entity is an entity to which, through by the grammatical form of the discourse employed in speaking of its existence is ascribed, yet in truth and reality existence is not meant to be ascribed (Bentham, 1997) », then it seems difficult to take the object of mathematics, whatever it be, for anything else than some fiction. Which does not mean that mathematics be without any object; « everything that is or can be done

in the way of algebra is referable » (Smith & Burston, 1983) said Bentham and he could have said it from arithmetic, geometry, topology and probabilities. However, those objects have neither existed by themselves nor have any place to be considered like magical objects; they appeared at the interlacing of a multiplicity of discourses, some holding and fixing what others are criticizing, working on, or qualifying. The same applies to the objects of mathematics as to the objects of language: on the one hand, language keeps using fictions and could not exist without them (Smith & Burston, 1983)[1]; on the other hand, language never ceases to stratify, and continues to transfer its objects, always becoming more complex, and creating ever higher levels of complexity. However, even though that stratification would make us think of some *superstructure* erected from a base or from a pedestal considered more real, we should be wary of this term used by Ferguson (1995, p. 159; p. 180) and, several times, by Bentham on his own, and we should not forget that no language is forever the foundation of all others, that each can become the foundation of another and that « taking for real » is not enough to confer reality (Cléro, 2021, pp. 370-371)[2]. The fact that a notion is not taken for real in an argument prejudges in no way that it cannot be in another. The notions taken as fictitious can be transformed and reversed in foundations, while those that are directly held as foundations may be transformed into fictions. When we talk of an object to be referred to by notions, we speak in a language [or in languages] of another language [or of other languages] that has [have] no objects more existing than his [their] own (Smith & Burston, 1983)[3]. Objects are imaginary

[1] *Chrestomathia*, p. 371-2: « To say that, in discourse, fictitious language ought never, on any occasion, to be employed, would be as much as to say that no discourse in the subject of which the operations, or affections, or other phenomena in the mind are included, ought never to be held: for no ideas being ever to be found in it which have not their origin in sense, matter is the only direct subject of any portion of verbal discourse; on the occasion and for the purpose of the discourse, the mind is all along considered and spoken of as if it were a mass of matter; and it is only in the way of fiction that when applied to any operation, or affection of the mind, anything that is said is either true or false ».

[2] We have encountered this problem before, in a collective work I led under the title *Les chemins du scepticisme en mathématiques*.

[3] *Chrestomathia*, p. 376 : « … for the explanation of them, language other than that in which these formulæ are delivered, may consequently be employed: other language, viz (-for there is no other) then language which is in common use ». Philosophy has the same problem, even though it only takes place in vernacular. For instance, if I want to question the existence of things in themselves, and if I defend the phenomenism, so I have to give, through a discourse, plausibility to things in themselves, even if I object to them by an other

mirages that appear in the intertwining of languages or, if you prefer, in their reciprocal and indefinite translation. There is neither object nor object position without translation; or transposition.

It would be wrong to believe that « mathematical objects » have a sort of existence thanks to the vernacular; considering that the forms of languages that are algebra, topology … have sense only from the vernacular foundation that would function as a set of real entities. There are beings and acts that have their sense only by algebra and that exist in an awkward and inadequate way, in the everyday language. It is not appropriate to have too ontological positions, as if an object-discourse could exist: a discourse in which things could be more easily spoken of, or in which things would be told more fundamentally. There is no object-language; Berkeley has shown that the ordinary speech is an imaginary speech that fallaciously refers to objects that seem real or delivers a speech that falsely pretends to be at the edge of those objects. That does not prevent, as we will see, the author from being very critical of scholarly discourses like the infinitesimal calculus that raises very small quantities before neglecting them, either in acts that follow immediately or in the object itself to which they pretend to be referred. Berkeley will accuse mathematics to be the main culprits of the sin of realism; realism being the illusion of the existence of the objects of which we talk in a descriptive behavior. For if mathematics suggests that there are objects apart from their discourse, how would perception, that seems more in touch with things, not give that illusion?

Nevertheless, we defend ourselves badly against the desire that makes us distinguish degrees of fiction or of reality in the mathematics discourses and that seems to highlight that there be multifarious degrees of proximity of those discourses to objects. Inverse problems seem less real and more fragile than direct problems, as if there were privileged direction and sense, and as if direct problems had more solid objects than inverse problems. By proposing such and such figure, we can show that the circle allows a smaller ratio of perimeter and area; but it is more difficult to show that the circle is, in all cases, the figure that provides the smallest ratio between the perimeter and the area it surrounds. Is this also true in spaces of any geometry and particularly in non-Euclidean spaces?[4] In the same way,

discourse that I hold in the same time and that carries the weight of the phenomenist criticism.

[4] We could have shown that: as what is « direct » seems always more real than what is « indirect », what is simple seems also more real than what is complex.

the « detachment » of the last lines of demonstrations that stand out from the other propositions leading to them seems to provide a greater complicity with things. But now Berkeley has shown -or he believes to have shown- that the illusion of what is detachable from our thought, and then participative of the phantasm of the object, was linguistic, if we dare say. The fact remains that the character « detachable » of the last propositions of demonstrations has extremely different modes and degrees that maintain the phantasm of detachment of objects of which mathematics is supposed to speak when they have achieved their demonstration.

Then the mathematical object has fictional status; but, among those fictions, some of them seem to be more fictitious than others in different ways. Fiction is certainly a relative notion: any notion can be taken for a fiction of another; yet some notions appear more fictitious than others. We must take a closer look at the notions that we hold in a privileged way to be fictions.

It seems that there are at least two modes of fiction: a *subject-mode* and an *object-mode*. Are relevant of the *subject-mode* the acts, the operations, the methods; and of the *object-mode* rather the results of the former. Without ignoring that a kind of back and forth circulates between these two modes, because we can enclose in an object-mode operations that are thus condensed; and what is given in an object-mode can also be the opportunity to carry out new operations that are not those that produced the objects.

A typical example of the subject-mode of fiction may be illustrated by the use of what Fermat called *adégalisation*: the same could have been said of the fluxion method or of the infinitesimal calculus method. The *adégalisation* is the operation by which you give a quantity an increase or a decrease that firstly plays a very effective role in a demonstration or in a calculation, and then, in a second time, will be suppressed because this increment or decrement, which has priced what you expected, is very small and very near from zero; so this operation goes through phases in which you made in the one exactly the opposite of what you made in the other. When a mathematician lets a magnitude « fluctuate », he does exactly the same thing: what allows him to neglect the difference between a very little fragment of a curve and a very little fragment of its chord, after having used their quasi-identity. That is what Berkeley will challenge to the calculators of his time -Newtonians or Leibnizians-: they give themselves the opportunity to carry out, on the same object, operations at one time contrary to those which will be carried out a moment later. To get results, contradiction creeps into mathematics with the extraordinary freedom that the decision-maker gives

himself. The *adégalisation* can produce objects that condense this « before » and this « after ».

We can indeed remove the « decision » and voluntary aspect of the operations and make with them objects that condense contradictory moments. So, the objects become themselves condensed products of contradictory operations: they take it upon themselves to be, at the right time, something or nothing. Such are the infinitesimals, the indivisibles, the quasi-zeros, the zeros of straight or curve lines (of areas, volumes, four or n-dimensional volumes, etc.). It follows that, to those fictions of « zeros of » correspond, in the same way, but inverse, the fictions of infinite sums of points which become straight lines and curves; of sums of straight lines and curves that become areas, of sums of areas that become volumes, of sums of volumes that become super-volumes. Those super-volumes have sense only in 4-dimensional spaces and thus open-up the possibility to the geometer to give himself n-dimensional spaces that were long treated as fictions, during the time that Pascal required to perform his operations on super-solids, before they can become no less real than the operations requiring only three dimensions. The register of infinitesimal calculus is inexhaustible in fictions: should still be mentioned the « accompanying curves » (that allow substituting areas that we know how to calculate them for surfaces we ignore it), the curves that run virtually inside buildings in order-to provide a set of equilibrium points of arches, vaults, cupolas, domes. We would find in the same way being that, though directly inadmissible as $\sqrt{-1}$, know how to slip away at the right time for solving equations that would have been insoluble otherwise.

There is certainly a difference between posing a contradictory being, posing a being the stability of which is indefinitely delayed [π], posing a counter intuitive being [$\sqrt{-1}$], authorizing oneself to do what one refrained from doing just a moment before or what we will refrain from doing just a moment later, acting with unverifiable notions; even though all these actions wrap themselves in the name *fictions*. In the same manner, probabilities enclose and concentrate contradictions: an event that can be and that cannot be, at the same time; with the resulting division between subject and object. When it is easier for the subject to assume contradiction than to entrust it to objects themselves, probability becomes a degree of certitude (or incertitude) of the subject who transforms his ignorance concerning an event into a degree of presumption or belief that it will be such or such; that it will be or not. In the probability calculus, the subject can carry the being of what has been, of what is, of what will be; the being of what is not, of what is no

more, of what perhaps will be, or ask objects to do it, under the French adjective of « stupéfiant »[5].

Another sense of fiction should not be forgotten either: the storytelling of a problem of mathematics or of a mathematical theory. A discourse of mathematics lacking in storytelling is unlikely to be accepted, heard and even « understood ». Without denying that the beauty of demonstrations has an influence on their truth, it is less here a question of beauty than of knowing whether the story has a tip, whether one is taken by it or not. When a mathematician takes you by the hand, he has something in common with Balzac or Stendhal who opens *Le lys dans la vallée* or *La Chartreuse de Parme*; with La Fontaine's way of leading you towards the moral or with Descartes inside his *Méditations de métaphysique*.

The most paradoxical point is that it is impossible, in multifarious cases, to get right propositions the truth of which is widely recognized by most of the community of mathematicians, without using fictions, the status of which is questionable from the very point of view of truth (Bentham, 1997, p. 371)[6]. We therefore obtain, in very large numbers, truths only through biases that are not true by themselves. Berkeley can do nothing about it. One can well challenge the « truth » of these calculations that resort to fictions, denounce those who prefer utility to truth, one cannot invent anything worthwhile without these fictions; the problem being to do with them -and with their accommodation conditions- rather than to reject them and never obtain anything but approximations.

A last word on this point: every effort that has been made to attempt a compilation of an exhaustive list of the kinds of fictions in mathematics (H. Vaihinger), as that to compile the kinds of demonstration styles (G.-G. Granger), has failed. It seems very difficult, if not impossible, to complete works of such kind; their reissues usually bring no improvement. Their list does not grow if even their author does consider that it is not doable. But these works leave traces. We must read these lists as metonymies of a work that could be more general and more complete. Of course, only small fragments are presented, but they let us dream of the rest as if the work were not undoable.

[5] In the *Adresse à la très illustre Académie parisienne de sciences*, « stupéfiant », stupefying, is the attribute chosen by Pascal to qualify his own work about the calculus of divisions.

[6] Bentham highlights this paradox, p. 371, on the special case of fluxions: « With this falsehood, how is it that mathematical truth, spoken of as truth by excellence, is compatible? ».

II. The discourses of fictions: a great diversity in their distinction, in their grouping and in their objects

Language is made up with propositions rather than with words. Very classically, Bentham distinguishes three kinds of discourses in mathematics: geometry, algebra, and the vernacular. One might be surprised at the bias towards simplicity: for instance, where to locate the topology that differs from geometry in dealing only with magnitudes, without measuring them by some quantitative value -what geometry will often look for on the side of algebra, as Leibniz blamed Descartes for it, in his *analysis situs* of which he presents the sketch to Huyghens-? But this way of dealing with topology (or rather with its ancestor, the *analysis* situs) highlights that Bentham distinguishes, through these three families of discourses, only functions or bundles of functions of which he simply analyzes in broad strokes the ways they relate each other.

Very classically, once again, Bentham sees the advantage of the algebraic bundle over the geometrical bundle in the immediate generalization that the former can perform, while the later, because of the need to draw figures and to reason from them, obtains its universality only by ecthesis and indirectly ((Bentham, 1997, p. 365-366).[7] The drawing implies that we give letters to points, to straight lines and to curves … and that we reason on these letters while taking care to take them always out of their particularity. It is as if we read algebra in putting real numbers in place of letters that directly ensure the generality. I may read as well $(2 + 2)^2 = 2^2 + 2(2 \times 2) + 2^2$ as $(a + b)^2 = a^2 + 2ab + b^2$; however the former formulation, though not false, would be regrettable in harming the immediate expression of universality, because of the very particularity of its expression; and in avoiding expressions however available. One can see an advantage to this

[7] « In the case of geometry, […] the language is particular, being by the want of general terms, confined, in respect of the subject, to the individual figures and parts of figures exhibited by the individual diagrams , and designated -not by any indication given of their intrinsic and permanent relations one to another, but- by the arbitrary and unexplanatory denomination given to them by means of so may combinations of the letters of the alphabet. In this case, one great instrument of elucidation, therefore, consists in the substitution of terms expressive of general ideas, being those of so many sorts of relation, to denominations thus individual and unexpressive. But in the case of algebra, the terms employed, abbreviated, and to those to whom the use of them is not familiar, obscure and perplexing, are as general as it would be in the power of words -of words at length and unabbreviated, to make them. For generalizing designation, in the character of a new and as yet unknown instrument of elucidation, no room is left in algebra. »

particular representation that geometry imposes on the mind in maintaining constant tension in the mind that must never relax its attention, while algebra, once the problem has been posed, once you have made sure the settings of the situation we intend to deal with are all there and all rightly weighted relative to each other, allows a mechanical and blind production of the result that draw attention to the signs rather than to an alleged object (Smith & Burston, 1983, pp. 374-375).[8] From this point of view, Bentham does not hesitate to say that geometry is more « tonic and invigorative » for the mind than algebra, because of the sustained intelligence demanded by the former (Smith & Burston, 1983, p. 369).

Bentham sees, between algebra and geometry, another discrepancy of which the importance in physics will grow over the ages and will have many repercussions in return in mathematics. Newton took pride that his geometry imitated the processes it staged, as if nascent and vanishing magnitudes that enter the reasonings were the direct image of appearing and disappearing phenomena. This imitation is of course an illusion and a sham advantage. Algebra, that imitates nothin (Smith & Burston, 1983, p. 358)[9] can advantageously account for phenomena it does not resemble and does not seek to resemble. The resemblance of geometry with pretended objects is a parasitic idea that algebra does not recognize. The lack of imitation, that seems the great default of algebra, is really a prodigious advantage upon geometry: the mental acts that explain a process have not to act out what they are trying to explain, as if through figures it were possible to give images in a mirror. The advantage of algebra upon geometry is expressed by a fine notation, though not perfectly (Smith & Burston, 1983, p. 370-371)[10];

[8] In *Chrestomathia*, Bentham quotes Lazare Carnot that explains that geometry implies reasonings and no unintelligent mechanics. Certainly a right use of algebra does not skimp on the preparation of calculations; but once this propaedeutics has been instituted, the algebra technician works in automatic mode, as C. Villani said in *Théorème vivant*. Bentham does hesitate to say that geometry may be considered as the intelligence of algebra. Let us compare, by way of example, the mechanical enumeration of all the developments of a situation in the calculus of probabilities to the Pascalian geometrical way to proceed that seems to be on one level with the thing itself.

[9] Unlike geometry and the other diagrammatic operations where « the signs employed were, or at least were endeavored to be, made analogous, i.e. naturally expressive »

[10] After having explained that there cannot be language without fiction, and that we must not confuse *fiction* with *fallacy*, Bentham weights, in *Chrestomathia*, as if we had choice, the advantage to express by fictions and compares it with the advantage not to resort to them. Moreover he believes he can express the idea that we have just enunciated as following:

Bentham thinks that all that is geometrical can, without any loss, be expressed in algebra, which retains only the essentials of relations, without bothering with anything unnecessary;[11] whereas the inverse convey, from algebra to geometry (the geometrical expression of all that could be said in algebra), involves a lot of losses; a loss of generalities, a needless detour by I don't know what needs to be seen, even though what we are speaking of could not be seen. Therefore, Bentham blames -perhaps he thinks of the *Essay*, yet so admirable of Bayes (1970)- that the demonstrations of the calculation of probabilities are still done, in his time, by the geometry of fluxions: what gives a false view of probability, that does not require it nor derive any benefit from it (Smith & Burston, 1983, p. 373)[12].

The huge interest of this Benthamian Appendix, which looks more like a construction site than like a finished job, is that its author turns our thoughts not only towards a conceptual confrontation of two groups of methods but also towards the conflict of two very different families of imaginary of which the interlacing and the oppositions in no way coincide with those of notions; nor with symbols that cluster, in their notations, into more cohesive wholes than it could be imagined as long as they are not studied. The very use of them provides no advantage if we don't look at them closely. Let's look at a detail that can be generalized.

An Anglo-saxon, that puts neither accent nor point on the letters of English language in its ordinary use, unlike what we can and must regularly put them on the letters of French, Spanish, Italian, on every vowel and consonant, like the Greeks and Latins, can, without risk of confusion, put points on some letters of his calculations. The English language marks the reiteration of the same operation upon what is already the product of that operation. So, speed, which is a ratio between space and time, can still be subject to a redundancy of that relation, in calculating the speed of the speed, which is acceleration, etc. A Francophone cannot mark this reintegration in the same way without encumbering his imagination whereas

translating a problem of algebra into a problem of geometry implies fiction; while the translation of a problem of geometry into a problem of algebra does not.

[11] What is the point of talking about triangles if, in euclidean geometry, I can need, with any loss in the operations I have to do, the property of the quality of their areas obtained by the half product of the base and the height?

[12] *Chrestomathia*, « But, if in a case in which figure cannot have place, as in the case of calculation concerning degrees of probability, as expressed by numbers, if any proposition be clothed in the geometrical form, so far will fiction have been employed, and with it, its never-failing accompaniment -obscurity, have been induced ».

the Anglophone can do it without taking this risk. The Francophone will immediately prefer the dx, ddx, dddx [d^2x, d^3x], or -for the inverse operations of summation- the ∫, ∬, ∭, ⨌… of which the Anglophone has not immediately felt the need. Of course, the Anglophones have adopted the Leibnizian system of notations and have abandoned the Newtonian system like the locutors of the other languages that judge it confuse; but we easily see that notation systems, whatever erudite they may be, retain some of their coherence because of their proximity or remoteness to the vernacular. If they are not exactly points, a dash or a double dash above the same letters may reappear curiously, without taking a sense openly Newtonian, in the equations of the game theory as written by J.C. Harsanyi. But they no less figure the reiteration of the notion of *utility* into *utility of utility*, or with still more evidence, the doubling of probability into *chance* to have reason to wait for a degree of *probability* in a situation -notions that do not shock a Bayesian in the calculations that require them (Harsanyi, 1976)[13]. According to the vernacular of each, the signs -particularly the game signs- do not have the same degree of ability to designate an object, if even they designate the same object.

So, the symbols of a system of signs may look arbitrary: they are not without reasons, would it be because of their necessary coordination (Smith & Burston, 1983, p. 372)[14]. The designation dx may well signify much the same as x with a point above it, or ddx, the same as x with two points above it, those notations do not follow the same logic. To be recorded, these gestures had to be made. This effectuation in a symbolic language -that can be called algebra- must have been possible by the binding of other languages, the vernacular in particular. No sign ever works in isolation; everyone functions in a system it constitutes with other signs; but also, with the signs of other systems which involve very different imaginaries, even though one would readily imagine that, whatever the symbol may be, it is

[13] One can read, from this point of view, chapter X of the *Essays on Ethics, Social Behavior, and Scientific Explanation* of J.C. Harsanyi that uses these notations in his equations.

[14] *Chrestomathia*, If, « antecedently to association, with a very few exceptions for the designation of anything which is to be signified, any one sign is as proper as another ». « But when associations have once been formed, this original indifference is at an end: for the designation of any object, some word or phrase should be looked out, which, in virtue of some meaning with which they have already been invested, serve in some measure to lead the mind to the conception of the thing meant to be designated, and in that respect are better adapted to the purpose than any words taken at random: than any word, in short, between which and the object which is to be designated, no such relation has place ».

possible that it relates to the same object, since this symbol is perfectly arbitrary. Really, however arbitrary, this arbitrariness is ever but relative.

There is not only one imaginary at work in a mathematical text; this one is always traversed by a group and contest of imaginaries. Like in literature, a novel, a poetry interlaces its imaginaries behind the symbolic of words, there are many imaginaries that interlace in sciences: the schemes -trees, arborescence, speeds, accelerations, urns, lotteries, etc.- are extremely important and they give impetus and diverse directions to the question dealt with. Words themselves that are used (root, power, square, cube, etc.), through the relationship between the signifiers and the signified, contain and intimate an imaginary that is very different from a vernacular to another.[15] Let's not believe that it can be set aside to the point of partition: those who were to reject all imaginary never succeed. It is no more possible to do mathematic without word than any other science -physics, biology, etc.-; but words inevitably peddle their imaginary. It is untrue that we could do without the words of *point*, *line*, *circle*, *volume*, etc.;[16] simply, they don't have the same sense by the pen of a Euclidean author, than under the author of the *analysis situs*'s one or in the books of Hilbert who conceive them so algebraically that those words indicate no more than relations, without their imaginary being supposed to interfere. These terms are not ordered in the same way: under the same names, the complexification order of the *analysis situs*'s beings is not the Euclidean geometry's one. We move them, that's all; and they tolerate it well.

Lastly, there is no uniformity of the discourses in their references to objects; it being understood that, in every case, the mathematic object, whatever it be, does exist, not in itself, but in the intercrossing of multifarious discourses. From a discourse to another, the object referred to is extremely changing. A problem of topology has not the same object than a problem of probability. However, problems that seem to have nothing in common may be able to benefit the same scheme: the arborescence, for example, is the same in the calculus of partition and in certain problems solved by Desargues. Moreover, besides those borrowings of schemes from

[15] The imaginary of the « milieu », for instance, is not the imaginary of the « middle » or of the « Mittel », despite the affinities; « zwischen » and « between », that are translated into French by « entre » do not refer to the same number : *two* in the former case; *three* in the later. In the symbolic language, « to be on », « to be between », « congruent » have no other meaning than to participate to the statement of axioms.

[16] Speaking, for instance, of *tables*, *chairs*, *bier glasses*, rather than *points*, *straight lines* (following Hilbert's quip).

very diverse parts of mathematics, the same scheme (of arborescence, for instance) may be used in two very different conceptions of the same sector: the one that works upon expectation -like in the Pascalian calculus of partition-, the other that deals with probabilities as one would arrange things -as in the case of Fermat-.

If we come back to our opposition between algebraic symbolism and the symbolism that, in geometry, grants more to the imaginary we can say that algebra has no point of view -what is not to say that it has no philosophy- whereas geometry requires necessarily a point of view, implying a located subject, even if location remains undetermined. We tend to draw figures in perspective; or, if we do not take action, we imagine them in perspective. Geometry, that places in position to « perceive », stages the particular that it asks constantly to overpass, to read transcendently. Geometry is always, to a certain extent, a more-or-less conscious topology. It is impossible for it not to adopt a point of view. Sometimes, things go far: this perspective is so well implied in geometry that the authors of its demonstrations ask the editors and printers to indicate shadows by streaks as if light could shine on figured objects from one side.

Anyway, there is a play between the interlacing of discourses and what they demonstrate. The ones seem to build scaffoldings around the object, providing a very diverse play of perspectives upon it: such is the case of the demonstrations used by Pascal to calculate centers of gravity of geometrical objects spawned by the roulette <cycloid>; the others seem to participate to the thing itself and be used as its logos: it could be that the same Pascal had made something like this in the calculus of partition, as if the mathematical discourse could serve as its voice. Algebra, that Pascal hardly practices, has the same gesture: it makes oneself the voice, not of things themselves but of relations between them. Still others take pleasure in a sort of point of view; it is precisely what Desargues does in his famous theorem. He uses the alignment of the points of intersection in order to establish a truth seeming above, that only the subject thinking the space can estimate because it seems to give arguments in favor of the three dimensions of the space (Hilbert, 2014)[17]. Therefore there are different possible postures of demonstrations about their objects. From this point of view, there is no

[17] Hilbert will put this point in question at the end of XIXth century in the *Grundlagen der Geometrie*.

difference between what happens in the uses of scholarly or symbolic languages and what happens in each vernacular[18].

There is a somewhat enigmatic word of Descartes about Desargues, whose the work he admired and whose generality of theorems he achieved thanks to his geometry that he qualifies *metaphysical*, as was to his eyes, Archimedes' work and his own.[19] What does this author mean, when we know that he was, with Fermat and Leibniz, one of the more algebraic mathematician? Does he mean that he reaches results that touch on the most metaphysical qualities of mind and matter? As, for example, in the famous theorem that, linked since 1638 to the name of Desargues, seems to achieve the certainty that space has three dimensions. Does he mean that he uses in geometry only what is strictly useful in his demonstrations? In any case, this word of *metaphysicians* addressed in a laudatory way by many thinkers to some mathematicians is itself under the pen of Bentham (Smith & Burston, 1983, p. 374)[20]. Does he not mean that there are mathematicians whose work requires metaphysical notions as it is the case, for example, in the solving of the *Essay* of Bayes that requires the staging of a subject that indulges in *expectations*? Is a metaphysical mathematics not a sort of mathematics that roots -to paraphrase a Cartesian image- deeply its discourse in notions that mathematics have in common with metaphysic? Can it not be a mathematic conscious and worried, in its work itself, of the staging of the subject and object in a metaphysical truth that would not be brought from outside?

It is right that some mathematics are more philosophical than others and some mathematicians more philosophers than others? Is it not the proof that bundles of language's uses in mathematics are closer than others of

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18 We have shown elsewhere how the English language had a very different posture towards objects that French language speaks of. The Anglophone has not towards the passive voice the repulsion that feels the Francophone who blames it as a heavy and harsh voice to which the active voice should be systematically preferred. The English language is more in a reflexive relation to the objects than the French that takes a more direct position toward the objects of the world.

19 In his letter to Mersenne of January 9th, 1639, Descartes wrote: « La façon dont il commence son raisonnement, en l'appliquant tout ensemble aux lignes droites et aux courbes, est d'autant plus belle qu'elle est générale et semble être prise de ce que j'ai coutume de nommer la métaphysique de la géométrie, qui est une science dont je n'ai point remarqué qu'aucun autre se soit servi, sinon Archimède ».

20 In *Chrestomathia*, Bentham notes the expression in Euler that, borrowing it to Carnot, amends it in « logical view »; what dulls it and perhaps deprives it of a part of its meaning.

bundles of language's uses in philosophy? Anyway, no language game is comparable to another in order to install some object required by a problem.

We have one more point to consider for understanding the ultimate fiction that allows to bind together the three discourses we have distinguished in following the vocabulary of Bentham, avoiding the awkward notion of *superstructure* and without resorting to the idea that such or such language would be more fundamental than the two others. We do not believe that one of these languages can be used as a *Grund* <*ground*> for the two others, as philosophers have often believed to have established it; but then which structure should we use in order that languages match each other without anyone being able to claim to be the stronger base of the other two? This at the cost of many confusions: of the expression of body with ordinary language; of attribution to the ordinary language properties it cannot have; of the imaginary with a secondary and negligible category in comparison with the symbolic. Then it is necessary to look now at that difficulty.

III. How is the triplicity of essential languages in mathematics articulated? A structure of Borromean rings

Borromean rings are not linked together like the keys of a bunch are bound with the ring that holds them: for if any one ring -and only one- breaks, all the others are released; whereas when the keys are commonly linked, if the ring of a key is broken, only this one is released, leaving the others bound together. Now, we must deal with the articulations of discourses that are of this Borromean order in their reciprocal dependence. They all assume each other -but in such a way that no one can be the base of the others without each other - is at the same time the base of it. In mathematics, the discourses are as competitive as concurrent.

Whereas it was always clear for Bentham that we could not do mathematics by a single discourse, but only in composing and combining their different modes (Smith & Burston, 1983, p. 362)[21] he varied greatly in the designation of the discourses that seem to collaborate to some work in mathematics and he does not forget joining to them a non-linguistic mode that does proceed neither in words, nor in symbols and images, but that is a mode of thought more dynamic and affective than representative. With variants, Bentham speaks of the *vernacular language* -without which we cannot do, even in mathematics that it would neither be possible to teach nor to

[21] *Chrestomathia*, we can use « neither the purely diagrammatic mode in any part, nor the purely verbal mode of expression, except here or there are or can be employed ».

learn-; of *symbolic languages*, like *algebra*, that he qualifies sometimes as *algorithmic*; of *geometry* that he qualifies as *diagrammatic* and that he puts aside for the reason it wants to intuit its spaces and objects into these spaces, had they more than three dimensions; lastly, then, beside these three discourses that are always being under different names and following the needs, a sort of affective structuring scheme to which the other discourses address, suppose and cultivate.

Let us take the example of Bentham to highlight the contest and the combination of the discourses needed to cross the « donkey bridge » of the demonstration of the Pythagorean theorem establishing that the square of the hypotenuse of a right-angled triangle is equal to the sum of squares of the sides of the right angle. There are many ways to demonstrate this theorem. We can, like Euclid, first build geometrically squares on each side, draw the relevant straight lines that allow to break the new figure into parts that, recollected together, allow to establish *in fine* the equality of the square built from the hypotenuse with the sum of many other areas skillfully recomposed in such a way that they appear equal to the sum of the two surfaces built from the two sides of the right angle. In this demonstration there is no spontaneous intuition; but we have painfully to build the adventive lines, to progress across constructions that seem dark up to the moment when we perceive late the sense of them, in overpassing their artifices (Miller, 1977)[22].

I can also, more intuitively and without building a complicated figure, mentally construct the square of which side is a + b and highlight that the right-angled triangles arranged on the sides of this square, but this time

[22] In the Preface of the *Phenomenology of the Spirit*, ten years before the publication of *Chrestomathia*, Hegel blamed in nearly the same terms as the author of *Chrestomathia* (p. 357-8) used to castigate the way for Greek mathematicians to demonstrate. Referring explicitly to the Pytagorean theorem, Hegel remarks that, in its demonstration by Euclid, «the nature of the right-angled triangle does not divide itself into parts in just the way set forth in the construction necessary for the proof of the proposition that expresses its ratio. The way and the means by which the result is brought forth belong entirely to the cognitive process. […] In mathematical cognition, insight is an activity external to the thing; it follows that the true thing is altered by it. The means employed, construction and proof, no doubt contain true propositions, but it must none the less be said that the content is false. In the above example, the triangle is dismembered, and its parts consigned to other figures, whose origin is allowed by the construction upon the triangle. Only at the end is the triangle we are actually dealing with reinstated. During the procedure it was lost to view, appearing only in fragments belonging to other figures». The mathematical work is nothing but a conceptual shift, a sort of metaphor.

turned inwards, make visible a square of which the sides are the hypotenuses of those right-angled triangles. I can then easily, in skillfully arranging the right-angled triangles, show that they are obtained by the square constructed upon the hypotenuses is equal to the sum of the square built upon the first side with the square built on the second. We deal with a puzzle that ends with the perfect fit of all the pieces.

That is difficult to utter in the vernacular, though it is easy to intuit the solution. This intuition can be perfected by another artefact, that was not conceivable by Euclid but that proceeds by algebra in using the remarkable identity

$(a + b)^2 = a^2 + b^2 + 2ab$

or: $a^2 + b^2 = (a + b)^2 - 2ab$.

We say exactly the same thing as what was intuited in the precedent game of squares, but we get a result that is not linked to a figure and that immediately gains more generality. We grasp in very few signs, without having to explain ourselves in the vernacular but with a strong symbolic help, that the Pythagorean theorem is right.

However, intuition is not to despise: it has its own rhythm, force and logic that cannot be expressed without any effort, by signs or by images, but through which we know with precision that it could be. Here are our languages which, in turn, are *concurrent* when they are not convergent. They are concurrent when their methods are different and when the result is more easily obtained by one than by the others. But they look to be common for the one who understands all of them and who knows how to spot through 2ab the area of four right-angled triangles. Then they are also *convergent* by the need to jointly resort to geometry and to algebra for solving some problems (Smith & Burston, 1983, p. 392)[23]. The « donkey bridge » is solved more easily by algebra than by geometry, but the former is strongly helped by the fact that the later can be underlying it.

[23] *Chrestomathia*, « In speaking of geometry and algebra […] -thus far it has been necessary to speak of these two objects, as if they were so many distinct branches of mathematical art and science; one of them, and that alone, applicable to one sort of subject or occasion; another and that alone, to another. But to his surprise not improbably, and to his no small annoyance certainly, the learner will sooner or later have occasion to observe, that in point of practice, no such separation has place; and that, for the obtaining of one and the same result, for the solution of one and the same problem, for the finding an answer to one and the same question, for the demonstrating the truth of one and the same assertion, […] both these branches of mathematical art and science have been employed at once ».

Those languages, far from being compiled as would be strata or foliations, are articulated and essentially dependent on each other.

Bentham points out that some ways to do mathematics are not understandable alone; that, in order to understand them, a help must be brought either to algebra by the vernacular, if we do not want to take -a x -a = a^2 for just a convention, but search to conceive the formula; or provided by algebra to the vernacular, would it be only by the means of the humble remarkable identities, easier to be understood in the symbolic language than in the usual one.

Let's begin with the paradox developed in *Chrestomathia*, p. 377, that poses the scandalous equality: -a x -a = a x a. The scandal -that Bentham is not alone to feel- (Stendhal, 1968; Smith & Burston, 1983, p. 433)[24] lies in the following alternative: either, in the highly rational discipline that is called *algebra*, we learn this kind of formulae like incomprehensible formulas from the catechism of some revealed religion; or we are trying to understand this expression, but then, what is possible only in the ordinary language and the least we can say is that the way is narrow. How is it possible that, in multiplying two negative quantities the one by the other, we obtain a positive quantity? How is it possible that, in the Cartesian coordinate graph, in inscribing the product -a x -a as a square area located in the lower left quarter, while the area a x a will be in the upper right quarter, we obtain yet, for these two opposed areas, the same result or a result noted in the same way? Understand how an algebra can obstructs its own intelligence!

We can manipulate these signs and use them as an arbitrary code, but if I want to understand what I write, I have to resort to the fiction, I can produce in the vernacular, of a debt that is withdrawn and so erased (p. 373-4) or, more generally, to the fiction of the removal of a removal, which is equivalent to a position (p. 379). The acts, the one that poses -a and the one that multiplies this -a by another -a, do not stack: one has an effect on the other otherwise adding itself to the other; it acts on the other by reflecting the other. If this remark better explains why -(-a) is equal to +a it explains however only one part of the difficulty that affects -a x -a. We have then to resort to the idea that: if multiplying by -1 restores the primary situation,

[24] Stendhal, in *La vie d'Henry Brulard*, chapter XXXIII, written in January 1836, twenty years after the publication of Chrestomathia, will tell how, studying mathematics with Chabert and Dupuis, at Grenoble, he was scandalized by this algebraic operation in a world where he was waiting for less hypocrisy than elsewhere. He tells of his dismay and he relates he was not convinced at all by the argument of the debt and its erasing. For more details, see the note 1, p. 433 of our version in French of *Chrestomathia*.

multiplying by any other negative number can increase or decrease (following the number is an integer or a fraction), contrary to what we imagine of a multiplication we readily but fallaciously identify with an addition (p. 380). We have to distinguish in the operation -a x -a, the multiplication a x a from the multiplication -1 (-a) , and combine them; what comes back to -(-aa), that is well equal to a^2. The idea, however subject to variations, is always the same: it is necessary to keep the connection of algebra with the vernacular -the language of words-. If we do not keep it, algebra becomes absurd and reluctant (p. 373). Bentham suggests, in a very subtle way, that some languages are better able than others to explain those paradoxes of algebra: he remarks that it is advantageous to read Euler in an translation into English rather than in a translation into French if not in German language in which the text was invented, for the English gerund allows to write: « the taking of a negative quantity negatively destroys the very property of negation, and is the conversion of negative into positive numbers » (p. 379); what the other languages cannot easily say.

But, as we noticed, the advantage of the vernacular when it comes to making sense of symbolic language, like algebra, is relative, because it is this vernacular that may, in turn, find itself in difficulty if it has not the help of algebra. The remarkable identities are a testimony of it and enough of a proof that the vernacular cannot support itself ; that it can be itself also an obstacle to its own understanding (Smith & Burston, 1983, p. 377)[25]. When, for instance, I read : $(a + b)^2 = a^2 + b^2 + 2ab$, or: $(a + b)(a + b) = a^2 + b^2 + 2ab$, I understand at once the detail of the acts; I forget none: the acts seem to me there, in front of me, and it is enough that I carry them out. On the contrary, if I have to describe them in the vernacular, certainly I can say that the square of a sum of terms is equal to the sum of squares of those terms to which we have to add the double product of the two terms. The addition seems, however, aberrant; would it be because it breaks the elegance of the formula that, for the pleasure of the ear, could have been interrupted like that: « the square of a sum of terms is equal to the sum of squares of those terms » and would have been wrong because of the oversight of 2ab that the so neat acts of multiplications in algebra make nearly impossible (Luce &

[25] It takes very little for making a formula incomprehensible or for shifting a right understanding. *Chrestomathia*,: « Placed on the threshold of the science, upon crossing the track of it, a little verbal inaccuracy, which, to the eyes and feet of an adept standing in the higher regions, will, like a threat of gossamer, be an object altogether imperceptible, will in the eye of many a learner, be, if not an insurmontable bar, a troublesome, and, for a long time, a disheartening, stumbling-block ».

Jessop, 1951)[26]. Of course, it is always possible, afterwards, once the algebraic formula understood, to express in usual terms what it means, without forgetting the lest of its constituents, but it is also no less certain that, if we had started from the vernacular to understand the operation of which algebra makes such a complete and luminous review, we would have missed it almost for sure in this usual language that makes the expression of that expression most intelligible, whereas it goes without saying for someone who is familiar with algebra. Without any doubt, the symbolic language, like the vernacular, had to be acquired; it is clear that these two languages support each other and that the one who masters the symbolic language better and better does not speak the same vernacular than the one who has not such a mastery.

When we have understood that no language stands by itself and that it must rely on another or others to understand itself, that each one can be considered as the scheme or the intermediary another, indefinitely, that each has its own images, translatable in others, however never completely, never perfectly, but leaving voids or dreaming of overlays, we do not want to take one for the foundation of all the others. Or, if we prefer, we can, at leisure, take one, alternately and in different circumstances, for diverse needs, as the foundation of all the others, following the structure of the binding of the Borromean rings of which such or such may be the foundation of any other ring, but also, at the same time, be founded by any other. The intuitive seeps between the symbolic and the vernacular. The symbolic gives to conceive to the vernacular. The vernacular « understands » the symbolic (te game of new signs) and the intuitive (the game of forces); or rather, the « symbolic » and the « intuitive » are understood by the vernacular. There are elations of reciprocal seeping between terms that are not confused but that are strongly individualized. A whole book would be necessary to make sure how the three linguistic forms are associated, starting with each one, *taken as* base of the two others; let's only take a few steps in that direction by reflecting on a singularly litigious case.

[26] Not quite yet, because Berkeley will show that it is a quantity of that kind that is « forgotten » or rather neglected by the Newtonian calculators in order to obtain their increases and decreases.

IV. The singular status of the vernacular

The vernacular is well known to be closer to intuition than the symbolic language; and we would say of it that it provides ground for the other languages. But this discourse proves inaccurate when it is considered more closely. We just saw that this discourse is not relevant when we observe the relationship between algebra and the vernacular: If we immediately understand that the terms are not handled in the same way in the parentheses or outside of them, it is certainly not by the vernacular that we know it. It took an insight that concerns the symbolic itself; the vernacular simply came to reinforce and confirm the insight. It is through the intuition of symbolic that we can write in the terms of the vernacular an algebraic equation, even very simple, but already too complicated for it. Not the other way around.

But one will object that the case of algebra is particular and cannot be valid for all the forms of the symbolic discourses. In the *Phenomenology of perception*, M. Merleau-Ponty thought he could show that the figures drawn by the geometer implied as much a relation to the living body as a relation to concepts; that we do not draw on the sheet the outline of the triangle supposed to help us solve the problem without this figure being arranged in a privileged way in relation to my body and to the world around myself; that, at least, one of its sides should be approximatively parallel to the ground; this parallelism and this ground being perfectly imaginary, though quite relative to the situation of my body. Then the geometrical figures of geometry follow a double logic: one, of the concepts involved in the problem to be solved; the other, less visible, more implicit -although it was necessary to acquire that « implicit »-, that is one of the presentation of the figures matching with the insertion of my body in the world. So, a student that would draw crooked a right-angled triangle, without one of its sides -preferably the hypotenuse- be substantially parallel to the ground, passes for a poorly educated boy, perhaps mentally unbalanced, in any case for ill-intentioned with regard to discipline. At the very moment that the solving of the problem (of the square of the hypotenuse) would not suffer from another presentation than the « statutory » presentation of the right-angled triangle because the squares built upon the sides seem to twirl around the figure. This « statutory » presentation, linked to the interference of the body in geometry, is not without a convention that risks intimidating misconceptions. Those figures stood up (relatively) to our body or to the image we have of it constrain the problem into a meaning that is not conceptually inherent to it, and they stand in the way of their solution even

in the issues where this meaning seems relevant. So, the figures that Pascal uses in order to calculate what he calls the centers of gravity of figures rotated from the cycloid stand as if they were to be placed on an imaginary ground or suspended from an imaginary fixation; but they have the disadvantage of confusing the reader's head. This confusion is not necessarily committed by geometers, but they are responsible of it by this presentation that gives too much to the imaginary of *my own body* and thereby entertains the confusion between the center of gravity that refers to the earth or to the center of the earth and the barycenter that owes nothing to them.

However, one would say, is it not you who are confusing the logic of the body with the logic of the vernacular? We answer that, without speaking of the identity of those two logics, we must, at least, underline their complicity. In his *Phenomenology of perception*, Merleau-Ponty (2002) does it in these terms: « If the words 'on', 'through', etc., are to retain any meaning, it is in virtue of my working on a perceptible or imaginary triangle, that is to say, one which is at least potentially located in my perceptual field, orientated in relation to 'up' and 'down', 'right' and 'left', or again, as we pointed out earlier, implied in my general grip upon the world. The construction makes explicit the possibilities of the triangle, considered not in the light of its definition and as a pure idea, but as a configuration and as the pole towards which my movements are directed. The conclusion follows of necessity from the hypothesis because, in the act of constructing, the geometer has already experienced the possibility of the transition» (Merleau-Ponty, 2002, pp. 448-449). Merleau-Ponty (2002) seems to find that, in the vernacular, close of the body language, the geometer can count on some help for his demonstrations. But more than a century before him, Bentham, far from taking for granted the thesis that the vernacular allows the understanding of all geometry, had insisted on the obstacle and the shift that the inevitable rooting of our ideas in the imaginary and metaphors of the vernacular, represent for the scientific conception of geometry and, more generally, of mathematics. Where Merleau-Ponty seems to be satisfied with some kind of agreement, certainly inevitable, between body, images, the language of body, the language of geometries, whatever they may be Hilbertian, Riemannian, and in any case non-Euclidean, Bentham highlighted *a contrario* the difficulties that the imaginaries of words, different in each usual language, oppose -without any rationality nor possible predictability- to conceptuality. Words do have their schematistic value towards concepts; their schemes (balance, tree, branches, roots, lotteries, urns, targets, speed, derivatives, fluxions, etc.) translate into each other, at the same time as concepts; but not

necessarily at the same rhythm as them; not necessarily by doing the same « leftovers » as them.

Doubtlessly, we can legitimately obtain that any discourse, technical, symbolic, geometrical, be expressed in the vernacular. Doubtlessly, Bentham marks his preference for the expression of the science in the vernacular, by pointing out that the latter is, among all the languages, the most malleable, the one that allows all the flexibility to the intelligent scholar that knows how to bend the tongue to what he wants to say and to accommodate his knowledge to all kinds of audiences he wishes to be understood (Smith & Burston, 1983, p. 393). If the scholarly and symbolic language tends to be unique, unlike the ordinary languages that allow all ways to resort to it, the ordinary language is the only one to be vehicular: it is its function (Smith & Burston, 1983, p. 355)[27]. It is only through it that the student can obtain the insight by which he will be able to seize a demonstration, then to appropriate it or some result in a symbolic language.

However, that does not mean that it is possible to invent the equivalent of symbolic truths, of their power, in the vernacular. In a noticeable article, J.-M. Lévy-Leblond has showed the failure of the physicists who believed they could present and even demonstrate results through bypassing mathematics and who pretended to give their equivalent to understand more immediately and more directly (Lévi-Leblond, 1985)[28]. The movement can go so far as to attempt to expose the mathematics striving to popularizing them into the vulgar language; but there can be no question about the creation or invention of anything scientific in it (Smith & Burston, 1983, p. 377), even to stick exclusively to it for teaching the very difficulties of knowledge. It is necessary for the symbolic language to be already there in order to straighten a vernacular, that would be, without it, a permanent error matrix. Vernacular languages are distinct from each other, and if one of them can help the symbolic language, on one point or on a very small segment of his course, because of its specific semantic and syntax,

[27] *Chrestomathia*: « Ordinary language is the sort of vehicle, and the only sort of vehicle, which is in possession of the employment of conveying ideas to the mind. In as far as any other sign, or set of signs, shares in this employment, -in as far as this function is performed by any special set of signs,- it is only through the medium of those ordinary signs».

[28] *Encyclopaedia Universalis* (1985). In this article, carried over under the title « Physique et mathématiques » into *Penser les mathématiques*, col. M. Loi & al. ed., Paris, Points Seuil, 1985, Lévi-Leblond castigates those who, at each stage of the development of the physics, were nostalgic for a less mathematical science, more directly expressed in the usual language, supposed more understanding.

it will become a main obstacle a little further. Bentham puts forward the vocabulary of powers and roots that can be useful to the understanding of the squaring or cubing or any raising to the power n, like to the inverse operation, of extracting square, cubic, n_{th} roots (Smith & Burston, 1983, p. 385). The inverse arrangement of these operations, which is like that of branches and roots of a tree, is suggestive; and it is visible that a correspondence between those linguistic schemes can corroborate a similar correspondence between concepts. But just up to a point. If the powers are like the branches of a tree, far from becoming bigger and bigger, the branches of a tree become more and more stunted; that verbal scheme of « powers » becoming incompatible with the concept. Likewise, the imaginary of square or cubic roots, if it seems to help the notions of *square root* and of *cubic root*, is quick to block the system on the other side, without any reason, in such a way that we are no longer allowed the thought of a number for which the division five times or more by the same number returns the first number by multiplication. So, the obstruction by the vernacular happens as much on the side of powers than on the side of roots (Smith & Burston, 1983, p. 387). The image induced by words -if it can ever be the same in all locutors- quickly becomes irrelevant; we must forget their meaning without dwelling on them if we want to speak conceptually with the vernacular. In the example chosen by Bentham, the obstacles are nearly the same in English and in French, but we saw that the English gerund gave English-speaking facilities to imagine subjective probabilities and particularly to have invented them, for it was in English that they were invented. However, it is true that, once invented, the subjective probabilities can be expressed in any language whatsoever. The game is then complex between the vernacular language and the geometrical and symbolic language; and, though the temptation be so great, the first cannot be considered as the foundations of the last, neither for invention nor for pedagogy where the vernacular remains auxiliary.

Lastly, if believing in the fundamental character of the vernacular is nothing but an illusion, that belief is no less essential; it gives the impression, doubtlessly false but perhaps necessary, of a sort of contact with things in themselves, fundamentally existing: the impression to relate to something that is not linguistic, whereas we have showed that the objectivity was linked to an intersection between multifarious intercrossing discourses. It would be however notoriously insufficient to confine the vernacular in the role of simple scientific popularization. Scientific popularization of science describes or pretends to describe the objects of science. Now the task of the vernacular in the scientific field is not only describing. It holds all kinds of

other functions: those of giving objects to the symbolic discourse; giving a kind of substance to them, even fallaciously; supporting otherwise acts invisible in the technical character of the symbolic. Nevertheless, so essential that might be the illusions of the vernacular, that gives no higher truth title over the other languages.

We do not deny that some literary schemes happen to be the same than those of mathematics; but it would be false to make advantage of holding the thesis, erroneous in our eyes, that ordinary language is the fundamental language of all other languages. Literature is a specific work upon words, as demanding as can be the work of the mathematician or of the scientist that operate as technicians on the signs of symbolic language. Psychoanalysts have subtly discovered in the works of Franz Kafka (Goian, 1964),[29] Henri Michaux or Pascal Quignard, and so many writers, the presence of complex surfaces, volumes, super-solids in a syntax and a semantic that remain the words' ones; the writing flipping for example the surface of the anorexic body, in the way of a Klein bottle. If technicians of symbolic were to take pride in being the only ones to produce complicated volumes, it would require presenting literary works that concretely and sensibly assure them that literature is able to produce such works. Bentham castigates the pedants of the symbolic (Smith & Burston, 1983, p. 393)[30]. But we must not forget either that it is only by a specific work upon the vernacular that the literary language accesses the symbolic; and we have not to bring back literature for all purpose to some production of the body language, whereas the body language is nothing but derived, as many others, from the languages' game. Although the one and the other proceed very differently, literature and mathematics offer comparable work. The writer invents, in a complex use of the vernacular, the best way to make understandable an object that is immediately incomprehensible and indescribable in that language. Literature has not so many ways of producing its object, as opposed to mathematics; it produces it in crossing several languages between them. Understanding is always translating. To translate is

[29] F. Goian has written a chapter untitled « Logique et topologie de l'anorexie. À partir de la bouteille de Klain », in a collective book to appear soon to the MJW Feditions under the title: *Essais sur l'usage de la topologie en psychanalyse*. This text presents fragments from Kafka's diary in which the author plays inside and outside his body in the same way that the volume of the Klein bottle turns on itself.

[30] In *Chrestomathia*, Bentham denounces the delight of some scholars to become a stranger to the common sense.

to engage in reversible operations, ultimately boundless. The ground is the result; but it is no more ultimate than the process itself.

We end this section by two partial conclusions. The *first* is that we hold a tendency to grant the vernacular many qualities that it does not have - particularly all the characteristics that seem to assure its virtues to build community among scholars, among non-specialists, between specialists and non-specialists; so that one might even wonder if it is not its ability to accommodate those imaginary aptitudes that makes its principal virtue. The *second* is in answer to a question that may have crossed the reader's mind several times: is there a criterion that should allow to designate -in this set of rings to which we have compared the diverse languages and that we described as *Borromean*- the language to be chosen in given circumstances? The problem becomes this one: if discourses are competing to express mathematical truths -as for all other truths-, if what is clear in the vernacular loses all clarity, and even all understanding, in the algebraic language, if, on the contrary, it may happen, in certain circumstances, that the algebraic language be infinitely clearer than the vernacular, if some languages seem to be more adapted or less adapted than others for some invention, then which will be the criterium allowing to choose to use one rather than the other? Should we let ourselves be tossed about from one criterion to another at the risk of being subject to all sorts of errors? But is that not asking for a criterium of the use of criteria? We know that it is when we need them the most that methods are usually lacking and that they let us down at the point where the methods should explain themselves the most: there is no method to use methods. No method may substitute itself to intelligence and judgment because intelligence and judgment are required to use the methods; as a result, we often discredit the methods and grant them only a mechanical character. Ultimately, there are no more criteria than methods because the very criteria must be chosen.

V. The unexpected advice of a singular apprenticeship

There is still one remark to be made; it is particularly important within the framework of a utilitarianism that demands an education and research both useful to the society. We need to perfect our system of Borromean rings that allows to renounce to the pretended models of foundation, and to promote the mathematic learning without using writing. What is the link between utilitarianism and the promotion of mathematics learned from memory?

When Bentham says that we must teach to students doing mathematics in one's head, without writing them and in mentally testing out a problem, even only in feeling its difficult points -that must be solved-, don't believe that it is a sudden concession to intuitionism, breaking with the assertions of algebraic symbolism, of the geometrical representation and of their link with the vernacular. These tenets are not far apart, and they are not abolished.

If I happened to recommend to young philosophers to read mathematics, pen in hand, to loyally understand the chain of reasons, in their abruptness, without just skimming over them,[31] I would notice without being alarmed, that Bentham explicitly gives, in *Chrestomathia*, to the students of mathematics -rather than philosophy, however-, the reverse advice: to learn mathematics in one's head. To tell the truth, we do it spontaneously, when we carry in our head a problem that we cannot solve on the spot, although we are seeking in all ways to solve it. The sophisticated extension of this spontaneous activity is essential and without any doubt a good method. It serves to mathematically, geometrically, think; and so, to calculate, in the utilitarian meaning of the term, getting a good intuition of what must be taken in account in a question stands, while remaining an intuition. The question is less psychology than utility.

Through intuition, we can grasp, without any speech, difficult problems, or such a moment of an arduous problem, retaining all its difficulties, even if some of them inevitably escape because we cannot solve it yet; or, when the problem is solved, supporting the point of a problem, without saying it. Intuition is not without symbolism; it cannot simply still express it at once; and it is only deprived of representation - more or less provisionally.

There is a time when the problem -particularly if it is difficult to solve it- must be carried on the mode of feeling, of the play of forces and reciprocal surpassing each other, that we represent neither figuratively nor symbolically. Which does not mean that this feeling lacks objectivity or universality. When Pascal says that the heart feels the three dimensions of space, he does not necessarily want to tell that the feeling is equivalent to the knowledge that space really has three dimensions and that it puts an end to the doubt of its three-dimensionality. It holds the tension; that is not the equal of the knowledge of what they consist of.

[31] That seemed and still seems to me that it is the only way to grasp the mathematical rigor, particularly from the reader unable to create it.

There is a time when tensions must be felt without it being necessary to represent them; when their issues are potentially kept without it being yet possible to know which will be chosen: to represent them would disturb the operation, scramble the forces implied in it and it only required to feel. Their best expression is yet to be discovered. Pascal said that the proofs of the existence of God are not wrong but that they are difficult to hold for more than an hour (Pascal, 1985)[32], synthesizing all the reasons in a single « affect ».

This education of mathematics learned in one's head is, evidently, of psychological significance; provided that one gives the term *psychological* a value that might not only be empirical, but also transcendental. To initiate oneself to the time of mathematization, of formalization, of problematization (forgetting the less possible parameters, except those for which we have weighted it was possible to neglect them without inconvenience) is to learn to pose all kinds of problems, to attempt all sorts of solutions to them, without having yet written nor exposed anything about them. For, if we have seen, there is a semantic and syntactic play of diverse discourses, there is also a play between writing and orality. Symbolism is necessarily written and can only be understood by writing.

The vernacular, when used in mathematics, is most often oral. We explain ourselves pedagogically in the vernacular. But, once again, the vernacular should not be systematically held to be more fundamental than the symbolic: the symbolic is so abrupt, compared with ordinary language, always about to confuse everything, that it leaves speechless in its immediate perfection; even if this stupefaction is not without intelligence.

VI. Conclusion

We will have noticed that we stuck to classical mathematics for our purpose, contenting ourselves with brief forays into more recent times. This is not at all, on our part, a way of suggesting that the problems we have posed relate only to the XVIIth and XVIIIth centuries; but this classical age provides us with examples that are faster to process.

What Bentham sees useful in mathematics, is a learning of linguistic styles operations. It is not necessarily a question of making everyone a mathematician; but it is that what we teach him be useful as much for what he will have do for the happiness of the great number than for his own. Therefore, multifarious axes are crossing in mathematics: of general and

[32] Pascal B., *Pensées*, Laf. 190, Br. 543, Sel. 222.

singular; of the written and of the oral; of the figurative and of the verbal; of the vernacular verbal and of the symbolic verbal; of the vernacular written and of the written symbolic. All along these lines, the matter is to learn how to do transcriptions that are perhaps more visible in mathematics but that are not destined to remain confined to that discipline (Smith & Burston, 1983, pp. 359-360).

It could not be maintained that, in Appendix n° VIII, either the term *topology* is very frequent, compared with the terms of *algebra* or *geometry*, or the allusions to topology are very acute. And yet the huge and indefinite system of inter-translations of mathematical languages appears to us as having a certain topological dominant through the privileged structure of Borromean rings. The topologic circulates everywhere and it is the *metaphysic* of the whole system.

It seems that Merleau-Ponty who was a friend of Lacan's, was conscious of the Borromean structure of these language alliances. For, if he underlines that «the fact that formalization is always retrospective proves that it is never otherwise than apparently complete, and that formal thought feeds on intuitive thought» (Merleau-Ponty, 2002, p. 448), he adds a little further that « 'Real', i.e. perceived, triangles, do not necessarily have, for all eternity, angles the sum of which equals two right angles, if it is true that the space in which we live is no less amenable to non-Euclidean than to Euclidean geometry. Thus, there is no fundamental difference between the various modes of expression, and no privileged position can be granted to any of them on the alleged ground that it expresses a truth in itself» (Merleau-Ponty, 2002, p. 455). It is evident that the scattering of that truth allowed Merleau-Ponty to save his dogma of primacy of the perceived upon the symbolic, of the corporal triangle over the triangle as a set of relations of the symbolic discourse; dogma strongly compromised by the Borromean structure of discourses' alliances, which is yet perfectly considered by the author. However, it is necessary, not to hesitate on this point, but instead to decide between the fundamentalist discourse and the Borromean articulation of the discourses. The former is nothing but the illusion of the later.

The case of these three languages looks like the tale of the three rings of which Boccacio said in *The Decameron* (I, 3) that they had an equal title to be produced in witness of right; but « which of them is justified in so believing is a question which remains pendent*, encore in pendente, e ancor pende* » (Boccacio, 2004, p. 57).

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